A brief but important note on the product rule

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The product rule in the Australian Curriculum

The product rule refers to the derivative of the product of two functions expressed in terms of the functions and their derivatives. This result first naturally appears in the subject Mathematical Methods in the senior secondary Australian Curriculum (Australian Curriculum, Assessment and Reporting Authority [ACARA], n.d.b). In the curriculum content, it is mentioned by name only (Unit 3, Topic 1, ACMMM104). Elsewhere (Glossary, p. 6), detail is given in the form:

If $h(x) = f(x) \cdot g(x)$, then

$$h'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x) \tag{1}$$

or, in Leibniz notation,

$$\frac{d}{dx}(u \cdot v) = u\,\frac{dv}{dx} + \frac{du}{dx}\,v \tag{2}$$

presupposing, of course, that both u and v are functions of x.

In the Australian Capital Territory (ACT Board of Senior Secondary Studies, 2014, pp. 28, 43), Tasmania (Tasmanian Qualifications Authority, 2014, p. 21) and Western Australia (School Curriculum and Standards Authority, 2014, WACE, Mathematical Methods Year 12, pp. 9, 22), these statements of the product rule have been adopted without further commentary. Elsewhere, there are varying attempts at elaboration. The SACE Board of South Australia (2015, p. 29) refers simply to "differentiating functions of the form , using the product rule." In Queensland (Queensland Studies Authority, 2008, p. 14), a slight shift in notation is suggested by referring to rules for differentiation, including:

$$\frac{d}{dt}\left[f(t)g(t)\right] \quad \text{(product rule)} \tag{3}$$

At first, the Victorian Curriculum and Assessment Authority (2015, p. 73) refers to finding the derivative of $f(x) \times g(x)$, where for some reason f and g are restricted to polynomial functions, without naming the result as the product rule, but later does refer to the product rule by name (pp. 75–76).

Again, the Board of Studies, Teaching and Educational Studies New South Wales [BOSTES] (1982, p. 53) refers to theorems "which allow us to find the derivatives of complicated functions," and specifically cites equation (2) without naming the result as the product rule.

A pedagogical problem

Apart from the fact that if we are to cite Leibniz's wonderful notation in equation (2) then it would be equally important to acknowledge Lagrange's even more concise notation in equation (1) (Merzbach & Boyer, 2011; Stillwell, 2002), I would like to suggest that there is something worrisome about both equations (1) and (2) as they are expressed, for the following four reasons.

First, classroom experience over 18 years suggests to me that using consecutive letters of the alphabet f, g and h in equation (1) can be confusing for students meeting (differential) calculus for the first time, and more so because h is often used for another purpose when differentiating from first principles. Writing $f(x) = u \cdot v$, where u and v are both functions of x, removes this issue.

Second, when equation (2) is read out aloud it sounds cumbersome, and just a small change makes this expression flow more easily:

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \tag{4}$$

Third, it is a tad more poetic, and hence conceptually easier, to defer now to Lagrange's notation and to write:

$$f'(x) = u \cdot v' + v \cdot u' \tag{5}$$

Fourth, even though the symmetry here in equation (5) is quite striking, note how 'wrong' it is, and how 'wrong' each of these expressions for the product rule is. Well, of course they are not wrong *per se*; however, they are wrong in the sense that it would be inappropriate and unwise to present the product rule in any of these ways, especially to students who are encountering calculus for the first time. This is because, in these forms, the product rule appears to be disjoint from the quotient rule. Using language from Piaget, if you will, there is a disequilibrium or cognitive dissonance experienced by many students when they are dealing with the product and quotient rules. Or, using more straight-forward but perhaps less precise language, many students do not see the connection between the product rule and the quotient rule, seeing them as two apparently distinct formulae and learning or memorising them as such. The resulting increase in cognitive load will make it difficult for many students to solve related problems with cognitive content, that is, the exact kinds of non-trivial problems that you would like your students to be addressing (Sweller, Ayres & Kalyuga, 2011).

The quotient rule

To appreciate the point that is being made here, let us (re-)visit the quotient rule and the derivation of the product rule. The quotient rule relates the derivative of the quotient of two functions to the functions and their derivatives. The senior secondary Australian Curriculum (Australian Curriculum, Assessment and Reporting Authority [ACARA], n.d.b, Glossary, p. 6) presents the quotient rule in two forms.

If $h(x)=\dfrac{f(x)}{g(x)}$, then

$$h'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2} \tag{6}$$

or, if $f=\dfrac{u}{v}$, in Leibniz notation,

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\cdot\dfrac{du}{dv}-u\cdot\dfrac{dv}{dx}}{v^2} \tag{7}$$

Finally, deferring to Lagrange's notation, we have

$$f'=\frac{v\,u'-u\,v'}{v^2} \tag{8}$$

The results (6) and (7) can be readily obtained from the product rule, once, of course, the 'function of a function' rule has been met—simply replace v by v^{-1}. Alternatively, following a suggestion from SACE Board of South Australia (2015, p. 29), it is neat to apply the product rule to $f(x) = h(x).g(x)$ and rearrange to make $h'(x)$ the subject of the equation.

A comparison now of equation (1) with equation (6), equation (4) with equation (7), and equation (5) with equation (8), should highlight the 'disjoint' that I have argued above.

Derivation of the product rule

Now, having already encountered differentiation from first principles in any honest-to-goodness introduction to differential calculus, it is straightforward to see how the product rule itself is derived. Starting with $f(x) = u \cdot v$, we have, with some hand waving:

$$\begin{aligned} f'(x) &= \lim_{\delta x\to 0}\left\{\frac{(u+\delta u)\cdot(v+\delta v)-u\cdot v}{\delta x}\right\} \\ &= \lim_{\delta x\to 0}\left\{\frac{u\cdot v+u\cdot\delta v+v\cdot\delta u+\delta u\cdot\delta v-u\cdot v}{\delta x}\right\} \\ &= \lim_{\delta x\to 0}\left\{u\,\frac{\delta v}{\delta x}+v\,\frac{\delta u}{\delta x}+\frac{\delta u\cdot\delta v}{\delta x}\right\} \end{aligned} \tag{9}$$

which directly gives the product rule as shown in equations (4) and (5). In the second line (9) here, the distributive law has been applied to the expansion of a binomial product (perhaps prompted by the metacognitive image of 'crabclaws', or perhaps by applying the mnemonic *FOIL—First, Outside, Inside, Last*), a process usually first met in Year 9 (Australian Curriculum, Assessment and Reporting Authority, n.d.a, ACMNA213). However, the binomial product could just as well and easily be accomplished in another way (is there not a mnemonic FIOL, or PhIAL—Phirst, Inside, Antitheses, Last?), resulting in:

$$\begin{aligned} f'(x) &= \lim_{\delta x \to 0} \left\{ \frac{(u+\delta u)\cdot(v+\delta v) - u\cdot v}{\delta x} \right\} \\ &= \lim_{\delta x \to 0} \left\{ \frac{u\cdot v + v\cdot \delta u + u\cdot \delta v + \delta u\cdot \delta v - u\cdot v}{\delta x} \right\} \\ &= \lim_{\delta x \to 0} \left\{ v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \frac{\delta u\cdot \delta v}{\delta x} \right\} \end{aligned}$$

Taking limits, we now have the product rule, in Lagrange's notation, as:

$$f'(x) = vu' + uv' \tag{10}$$

At last, in equation (10) we have succinctness, neatness, symmetry, poetry (read it out aloud), and a clear connection with the quotient rule (Equation 8). My argument is that presenting and applying the product rule in this form will reduce cognitive load for a significant proportion of students when they are approaching solutions to many kinds of differentiation problems.

Ironically, it should be noted that on its formula sheet the School Curriculum and Standards Authority (2015) does get close to expressing the product rule in this form, even though its curriculum documents do not (School Curriculum and Standards Authority, 2014), which can only add to the confusion experienced by some students.

Feynman's product rule

Richard Feynman (1918–1988) was an American theoretical physicist and Nobel laureate. During the Second World War he completed his doctorate on the principle of least action in quantum mechanics, from which later came his formulation of the path integral and his powerful Feynman diagrams, and he worked on the Manhatten Project which produced the first atomic bomb. From 1945 until his death, Feynman enjoyed a remarkable academic career, placing great emphasis in and taking much delight from his teaching. He was acclaimed for his clarity, in his thinking and in his presentations, and for the way in which his explanations made the deep truths of physics accessible to his students. For an excellent biography of Feynman, see Gleick (1993); if teachers would like to introduce their students to Feynman, I would recommend Feynman (1985).

As an example of Feynman's alternative way of looking at the world, and as a bonus to our discussion of the product rule, we can follow Feynman (Feynman, Gottlieb & Leighton, 2013, pp. 39ff.), and, with the obvious caveats, write (10) in the quite pretty form:

$$f'(x) = uv\left(\frac{u'}{u} + \frac{v'}{v}\right)$$

or, slightly more succinctly:

$$f'(x) = f \cdot \left(\frac{u'}{u} + \frac{v'}{v}\right) \tag{11}$$

Again, if we can apply the product rule once, we can apply it two times or more. Hence, if $f(x) = u \cdot v \cdot w\ldots$, then

$$f'(x) = f \cdot \left(\frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w} + \ldots\right)$$

Finally, since $u^a = \underbrace{u \cdot u \cdot \ldots \cdot u}_{a\text{ times}}$, and since $\frac{dk}{dx} = 0$ when k is a constant, then, if

$$f(x) = k \cdot u^a \cdot v^b \cdot w^c \cdot \ldots$$

in general we can write

$$f'(x) = f \cdot \left(a \cdot \frac{u'}{u} + b \cdot \frac{v'}{v} + c \cdot \frac{w'}{w} + \ldots\right)$$

Applications of the product rule

Exercise for the reader 1

Use mathematical induction and Feynman's version of the product rule to prove the fundamental result from differential calculus:

$$\frac{d}{dx}x^n = n \cdot x^{n-1} \text{ where } n \text{ is a positive integer.}$$

Exercise for the reader 2

Use Feynman's version of the product rule to derive the quotient rule, expressing it in the neat and pretty form:

$$f'(x) = f \cdot \left(\frac{u'}{u} - \frac{v'}{v}\right)$$

Exam-type questions

Differentiate $\frac{x^3}{x+1}$ (BOSTES NSW, 2014a, p. 5, Q. 11c).

Differentiate $\frac{e^x \cdot \ln x}{x}$ (BOSTES NSW, 2014b, p. 6, Q. 11f).

Rocket acceleration

In senior secondary Specialist Mathematics differential equations met may involve two variables, and should include the modelling of motion through

examination of force and momentum (Australian Curriculum, Assessment and Reporting Authority, n.d.c, Unit 4, Topic 2). In particular, the equation $F = ma$ is met in various guises, albeit, from my experience, with m often taken to be constant. However, exploration of this equation when both the mass m and the acceleration a are variable is quite accessible to students in Specialist Mathematics. Take, for example, a rocket accelerating vertically relative to its initial position at a fixed point on the ground. To accelerate, the rocket burns fuel, and hence its mass is changing with respect to time t. A better understanding of force than $F = ma$ is that net force F is equal to the time rate change of momentum, which is really what Newton's second law of motion is saying. Hence, if p is the momentum, and since momentum is equal to the product of mass and velocity, we have:

$$F = \frac{dp}{dt} = \frac{d}{dt}(mv)$$

Application of the product rule gives

$$\begin{aligned} F &= v\,m' + m\,v' \\ &= v\,\frac{dm}{dt} + m\,\frac{dv}{dt} \\ &= v\,\frac{dm}{dt} + m\,a \end{aligned}$$

since the time rate change of velocity, $\frac{dv}{dt}$, is simply the acceleration a. Making acceleration a the subject of the equation, we now have:

$$a = \frac{F - v\frac{dm}{dt}}{m}$$

As the rocket is burning fuel to accelerate, it is losing mass, which means that m is becoming smaller and $\frac{dv}{dt}$ is negative. Therefore, on both counts, as fuel burns the rocket will accelerate at an increased rate.

Conclusion

The leap into the wonderful world of differential calculus can be daunting for many students, and hence it is important to ensure that the landing is as gentle as possible. When the product rule, for example, is met in the *Australian Curriculum: Mathematics*, sound pedagogy would suggest developing and presenting the result in a form that pays careful attention to cognitive load theory and metacognition, that is, in a form that is clear and concise, has a direct connection with related concepts, and is aesthetically pleasing. The underlying structure and beauty of the product rule may also be highlighted through enrichment that introduces students to Richard Feynman and his succinct approach to teaching.

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